Section 4.7

Applied Optimization

The Closed Interval Method

For a continuous function f on a closed interval $[a, b]$:

(1) Find the values of f at the **critical numbers** in (a, b).

(2) Find the values of f at the **endpoints** of the interval.

(3) The largest (smallest) value is the absolute max (min) value on $[a, b]$.

First Derivative Test

Let c be a critical number of a continuous function f.

(I) If f' changes from $+$ to $-$ at c, then c is a local maximum.

(II) If f' changes from $-$ to $+$ at c, then c is a local minimum.

(III) If f' does not change sign at c, then c is not a local extremum.

Second Derivative Test

Suppose f'' is continuous near c and $f'(c) = 0$.

(I) If $f''(c) < 0$, then c is a local maximum.

(II) If $f''(c) > 0$, then c is a local minimum.

(II) If $f''(c) = 0$, then the test is inconclusive.

Optimization Problems

Solving Optimization Problems

(1) Draw a diagram (if applicable) and fix notation.

(2) What are you attempting to optimize? (the "objective function")
What are the constraints?

(3) What are the constants and variables?
What are the domains for the variables?

(4) Use the constraints to rewrite the objective function in terms of a single variable.

(5) Use calculus to find the absolute minimum or maximum of the objective function on the appropriate domain.

Optimization Problems

Example 1: A piece of wire 10 cm long is bent into a rectangle. What dimensions produce the rectangle with maximum area? Link

Fix notation: Let x and y be the side lengths of the rectangle.

Objective function: Area $A = xy$.

Constraint: Perimeter is $2x + 2y = 10$. Simplify the objective function:

$$2x + 2y = 10 \qquad A = xy = x(5 - x) = 5x - x^2$$
$$y = 5 - x \qquad A'(x) = 5 - 2x$$

Critical number: $x = 5/2 = 2.5$.
The domain of x is $[0, 5]$, so we can use the Closed Interval Method:

$$A(0) = 0$$
$$A(2.5) = 6.25 \quad - \quad \text{Absolute maximum}$$
$$A(5) = 0$$

Solution: $x = y = 2.5$ — that is, the rectangle is a **square**.

Example 2: Find two non-negative real numbers x and y such that their sum is 120 and their product is as large as possible.

Solution: We are **maximizing** the product and are constrained by the sum. Let x and y be positive numbers where

$$x + y = 120 \qquad\qquad P = xy$$

Substituting $y = 120 - x$, we obtain a 1-variable function for the product

$$P(x) = x(120 - x) \qquad\qquad \text{Domain: } [0, 120]$$

where $y \geq 0$ implies $x \leq 120$. We use the Closed Interval Method to maximize product:

(I) Critical number $x = 60$ and $P(60) = 3,600$.

(II) Endpoints $P(0) = 0 = P(120)$.

(III) The maximum product is $3,600$ from the values $x = 60$ and $y = 60$.

Find two non-negative real numbers x and y such that their sum is 120 and their product is as large as possible.

$\Rightarrow$ Optimized by $P(x) = x(120 - x)$ on the domain $[0, 120]$.

What happens to the problem if we remove the qualification that the numbers be **non-negative**?

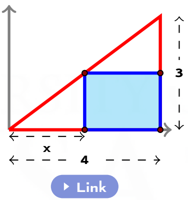
There is no longer a restricted domain. We are now maximizing on $(-\infty, \infty)$ and we cannot use the Closed Interval Method. Use instead the First Derivative Test:

- $P(x)$ increases on $(-\infty, 60)$.
- $P(x)$ decreases on $(60, \infty)$.

$(60, 60)$ is an absolute maximum point of P.

Example 3: What is the maximum area of a rectangle inscribed in a right triangle with side lengths 3 and 4? The sides of the rectangle are parallel to the legs of the triangle.

Solution: Fix notation: Let (x, y) be the top-left corner of the rectangle.



Objective function: Area, $A = y(4 - x)$

Constraint: The hypotenuse of the triangle $y = \frac{3}{4}x$

$$A(x) = \frac{3}{4}x(4 - x) \qquad \text{Domain: } [0, 4]$$

Critical Numbers in [0,4]: $x = 2$

Closed Interval Method:

$$A(0) = 0$$
$$A(2) = 3 \quad - \quad \text{Absolute maximum}$$
$$A(4) = 0$$

Solution: The maximum area is 3 units^2 with dimensions $2 \times \frac{3}{2}$

Example 4: A box with a square base and open top must have a volume of $32{,}000\,cm^3$. Find the dimensions of the box that minimize the amount of material used.

Solution: Let B be the length of the square base and H the height. We are minimizing the material (surface area) and the constraint is volume.

$$32000 = B^2 H \qquad\qquad S = B^2 + 4HB$$

Substituting using the constraint, optimize on the natural domain $(0,\infty)$.

$$S(B) = B^2 + \frac{128000}{B} \qquad\qquad S'(B) = \frac{2(B^3 - 64000)}{B^2}$$

Using the First Derivative Test, the critical number $B = 40$ is an absolute minimum. The dimensions of the box are $40 \times 40 \times 20$.

Example 5: An oil refinery is located on the north bank of a straight river that is 2 *km* wide. A pipeline is to be constructed from the refinery to storage tanks located on the south bank of the river 6 *km* east of the refinery.

The cost of laying pipe is $400,000 per *km* over land to a point P on the north bank and $800,000 per *km* under the river to the tanks. To minimize the cost of the pipeline, where should P be located?

Solution: If P is x kilometers to the west of the tanks, then $(6-x)\,km$ of pipes run overland and $\sqrt{x^2+4}\,km$ run under the river.

$$C(x) = (6-x)(4) + (\sqrt{x^2+4})(8) \qquad \text{Domain: } [0,6]$$

$$C'(x) = -4 + \frac{8x}{\sqrt{x^2+4}} \qquad C'(x) = 0 \;\Rightarrow\; x = \sqrt{4/3}$$

$$C(0) = 40 \qquad C\left(\sqrt{\frac{4}{3}}\right) \approx 37.9 \qquad C(6) \approx 50.6$$

The minimum cost is $3.79 million where P is located approximately 4.85 *km* west of the tanks.

Example 6: A landscape architect wishes to enclose a rectangular garden of area $1000\,m^2$. On one side will be a brick wall costing \$90 per linear meter and on the other three sides will be a metal fence costing \$30 per linear meter. What dimensions minimize the total cost?

Solution: We are **minimizing** the cost of the enclosure and are constrained by the area. Let x be length of the brick wall, which forms one side of the rectangle. Let y be the length of the other side of the rectangle.

$$1000 = xy \qquad\qquad C = 90x + 30(2y + x)$$

Solving for y in the constraint and substituting,

$$C(x) = 120x + \frac{60000}{x} \text{ with domain } (0,\infty) \qquad\qquad C'(x) = \frac{120(x^2 - 500)}{x^2}$$

The cost decreases on the interval $\left(0, \sqrt{500}\right)$ and increases on $\left(\sqrt{500}, \infty\right)$, so the absolute minimum occurs at $x = \sqrt{500}$.

$$x = 10\sqrt{5}\,m \qquad y = 20\sqrt{5}\,m \qquad \text{Cost } \approx \$5,367$$

Example 7: A poster of area $6000\,cm^2$ has blank margins of width $10\,cm$ on the top and bottom and $6\,cm$ on the sides. Find the dimensions of the poster that maximize the printed area.

Solution: We are **maximizing** the area of the printed region and are constrained by the area of the poster **and** the margin sizes. Let x be the width of the poster and y the height.

$$6000 = xy \qquad x \geq 12, y \geq 20 \qquad PA = (x-12)(y-20)$$

Solving for y in the constraint and substituting,

$$PA(x) = (x-12)\left(\frac{6000}{x} - 20\right) \qquad \text{Domain: } [12, 300]$$

We use the Closed Interval Method to optimize:

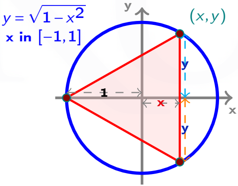
(I) $PA'(x) = \dfrac{-20(x-60)(x+60)}{x^2}$ and $PA(60) = 3840\,cm^2$.

(II) $PA(12) = 0\,cm^2$ and $PA(300) = 0\,cm^2$.

(III) A $60\,cm \times 100\,cm$ poster will maximize the printed area.

Example 8: Find the area of the largest isosceles triangle that can be inscribed in a circle of radius 1.

Solution: We are **maximizing** the area of an isosceles triangle and are constrained by the boundaries of a circle. The largest triangle will have each vertex on the circle $x^2 + y^2 = 1$. Put one vertex at $(-1,0)$ and the other vertices at points (x,y) and $(x,-y)$ where $y = \sqrt{1-x^2}$ on the domain $[0,1]$.



Link

$$A = \frac{1}{2}(1+x)(2y)$$

$$A'(x) = \frac{2x^2 + x - 1}{-\sqrt{1-x^2}}$$

Using the Closed Interval Method to optimize,

(I) 0.5 is the critical number in $[0,1]$ and $A(0.5) = \frac{3\sqrt{3}}{4} \approx 1.299$.

(II) $A(0) = 1$ and $A(1) = 0$.

(III) The largest area is approximately 1.299 units^2.

Example 9: Boat A leaves a dock at 1 PM and travels due north at a speed of $20\,km/h$. Boat B has been heading due west at $15\,km/h$ and reaches the same dock at 2 PM. How many minutes after 1 PM were the two boats closest together?

Solution: We are **minimizing** the distance between the boats and are constrained by the movement of the boats. Let the dock be located at $(0,0)$ and let (x_A, y_A) and (x_B, y_B) be the position of boats A and B, respectively.

$$(x_A, y_A) = (0, 20t) \qquad\qquad (x_B, y_B) = (15 - 15t, 0)$$

The distance between the points $d = \sqrt{(x_A - x_B)^2 + (y_A - y_B)^2}$ will be minimized when $D = (x_A - x_B)^2 + (y_A - y_B)^2$ is minimized.

$$D(t) = (15 - 15t)^2 + (20t)^2 \qquad \text{Domain: } [0,1]$$

Using the Closed Interval Method to optimize,

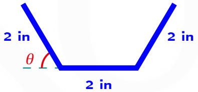
(I) The critical number $\frac{9}{25}$ has the distance $d\left(\frac{9}{25}\right) = 12\,km$.

(II) $d(0) = 15\,km$ and $d(1) = 20\,km$.

(III) After $\frac{9}{25}\,hr = 21.6$ minutes after 1 PM the boats were closest together.

Example 10: A rain gutter is to be constructed from a sheet of metal that is $6\,in$ wide by bending the $2\,in$ on each end upwards at an angle θ. Find the angle θ which will maximize the amount of water able to flow through the gutter.

Solution: We are **maximizing** the area of a trapezoid and are constrained by the width of the metal sheet. Let H be the height the ends rise and let B be the base of the triangle formed by the ends.



$$\begin{aligned} A &= 2H + 2\left(\tfrac{1}{2}BH\right) \\ &= 4\sin(\theta) + 4\sin(\theta)\cos(\theta) \end{aligned}$$

$$A'(\theta) = 4(2\cos(\theta) - 1)(\cos(\theta) + 1)$$

Since $0 \le \theta \le \frac{\pi}{2}$, we use the Closed Interval Method to optimize:

(I) $\theta = \frac{\pi}{3}$ is the only critical number and $A\left(\frac{\pi}{3}\right) = 3\sqrt{3} \approx 5.2\,in^2$

(II) $A(0) = 0\,in^2$ and $A\left(\frac{\pi}{2}\right) = 4\,in^2$.

(III) The angle $\frac{\pi}{3}$ or $60°$ will maximize the amount of water able to flow through the gutter.